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JUNE 1965

FUEL REQUIREMENTS FOR ATTITUDE CONTROL OF LARGE ORBITING APERTURES

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PREPARED FOR:

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum presents one aspect of a feasibility study of large orbiting apertures undertaken at RAND for the National Aeronautics and Space Administration, at the request of A. M. Andrus, to assist in determining requirements for the Applications Technology Satellite-4 program.

A large aperture established in a synchronous orbit would constitute a highly directional antenna. In order to use this directivity, it is necessary to have a precise attitude control system for the vehicle. This Memorandum provides certain working formulas for determining the attitude control fuel requirements for stabilization and control of large apertures in synchronous orbits.

A vehicle of this type is applicable to the fields of terrestrial communication, navigation and surveillance as well as space communication and radio astronomy. In addition, at such time as the vehicle size and weight, and mission requirements, are specified, the results of this Memorandum can be used to determine the weight which must be allocated to the attitude control system.

SUMMARY

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If a large aperture could be established in a synchronous orbit, it could be used as a highly directional antenna. In order to use these directional properties, a precise attitude control system is necessary.

This Memorandum analyzes the fuel requirements of the attitude control motors in order to achieve certain variations in orientation. The results indicate that the requirements for holding a steady attitude in either pitch or roll are reasonably modest, while maintenance of high-frequency oscillations of the vehicle may result in excessive fuel expenditure. The feasibility of any such system depends on the size of the orbiting vehicle and the attitude variations required for its mission.

Author

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LIST OF SYMBOLS

A_o	angular amplitude
a_y, b_y, c_y	direction cosines of Y axis in the x, y, z system
e	base of natural logs
f	driving frequency
g_o	gravitational acceleration
I	impulse
I_{sp}	specific impulse of fuel
I_x, I_y, I_z	roll, yaw and pitch moments of inertia of vehicle
i_o	orbital inclination
k_{1x}, k_{1y}, k_{1z}	proportional control constants
k_{2x}, k_{2y}, k_{2z}	rate control constants
M_{cx}, M_{cy}, M_{cz}	components of control moment
M_o	moment amplitude
m_F	mass of fuel required
m_o	mass of vehicle
n	acceleration in g's
p_o	solar radiation pressure
R_o	earth radius
r	radial distance from earth center to satellite
r_c	synchronous orbit radius
r_o	vehicle radius
Δr	center of mass - center of pressure offset
T	control thrust
T_o	response time constant

t	time
ΔV	velocity increment
X, Y, Z	orbital coordinate system
x, y, z	body coordinate system
α	pitch angle
α_i	required pitch angle
α_o	pitch amplitude
β	roll angle
β_i	required roll angle
β_o	roll amplitude
δ	phase angle
ϵ	orbital eccentricity
θ	orbital central angle
φ	phase angle
ψ	yaw angle
ω	driving frequency (rad/sec)
ω_n	undamped natural frequency of control system
ω_o	orbital angular rate
ω_x, ω_y, ω_z	components of vehicle angular velocity

I. INTRODUCTION

It has been suggested that if a large aperture could be established in a synchronous orbit, it would provide a highly directional antenna which could be useful in the fields of communication, navigation and surveillance satellites, as well as deep space communication and radio astronomy. In order to make use of the directional properties of such a vehicle, a precise attitude control system is required. Since the vehicle is fixed relative to the earth, its attitude can be determined from the ground by observing two crossed interferometers mounted in the plane of the reflector. The actual control of the attitude of the vehicle can then be achieved by means of low-thrust motors mounted on its periphery as shown in Fig. 1. Such a configuration could provide both positive and negative moments in pitch, roll and yaw by firing the appropriate diametrically opposite pairs on command from the ground.

This Memorandum presents an analysis of the fuel requirements for such an attitude control system for various modes of operation of the vehicle and represents one part of a feasibility study of this type of satellite vehicle. Since it is not the purpose of this Memorandum to analyze a specific satellite configuration, the fuel requirements are determined as a percentage of the total weight on orbit as a function of such design parameters as radius of the plate, specific impulse of the fuel, time of operation and certain characteristics of the required variations in attitude. Thus, the resulting formulas should be useful in arriving at the design of such a satellite.

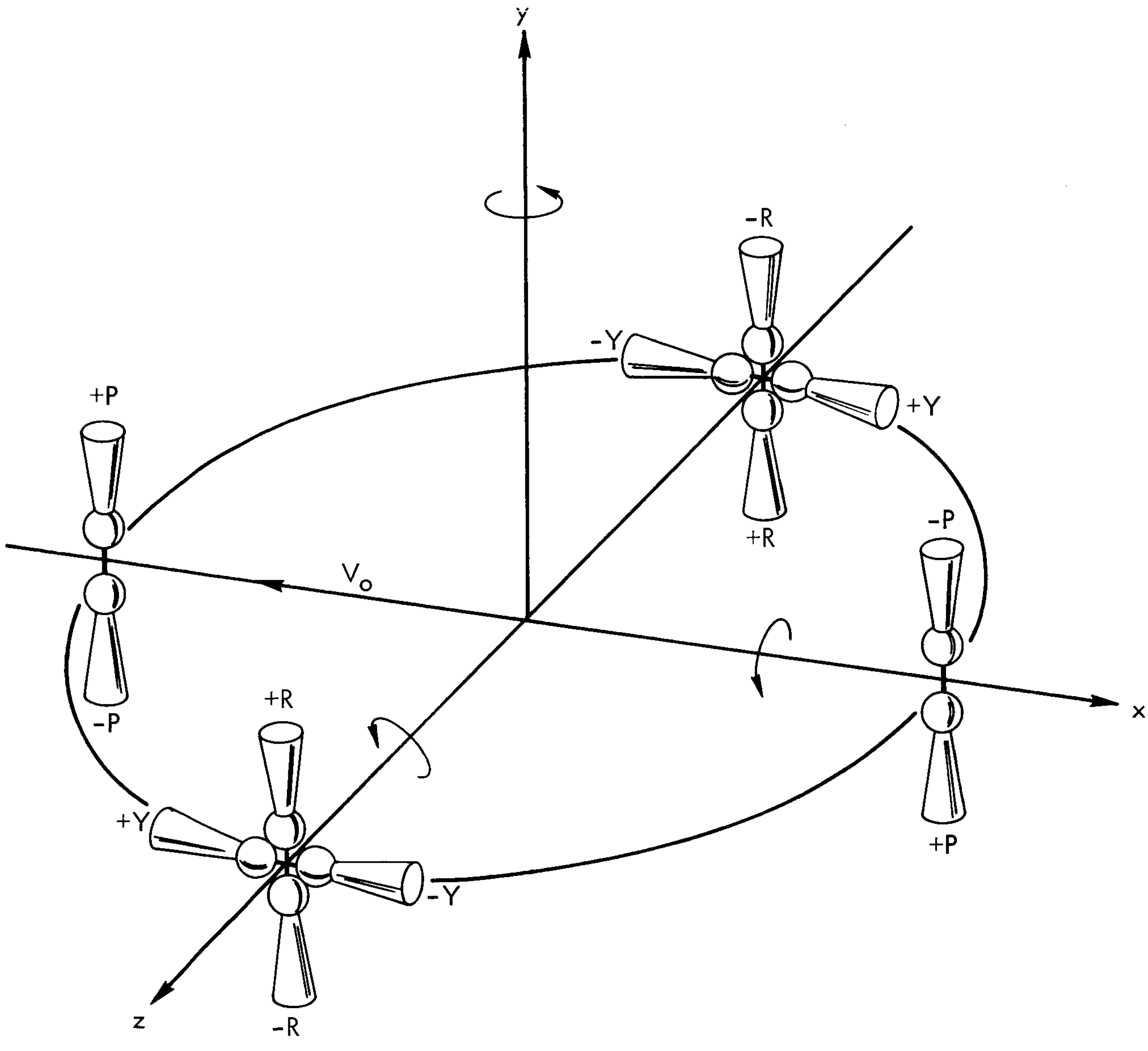

Fig. 1 — Control motor configuration

II. ANALYSIS

STATEMENT OF THE PROBLEM

For the purposes of this problem, the satellite vehicle is assumed to be on a circular synchronous orbit. A reference configuration of the vehicle is taken as a flat circular plate of radius r_o and mass m_o. To specify the orientation of the vehicle two coordinate systems are used, as shown in Fig. 2.

The X, Y, Z system is an orbital coordinate system with the Y axis along the instantaneous vertical and the Z axis along the normal to the orbital plane. For the circular orbit, the orbital velocity is along the negative X axis, and the orbital angular rate $\dot{\theta}$ is along the positive Z axis.

The x, y, z system is a set of body axes fixed to the vehicle with the flat plate in the xz plane. The orientation angles of this system relative to the X, Y, Z system are shown in Fig. 2 where α (pitch) is a rotation about the z axis, β (roll) is a rotation about the x axis and ψ (yaw) is a rotation about the y axis. Thus, for zero pitch, roll and yaw, the flat plate is oriented horizontally.

The problem is to specify a suitable control law for the associated control motors and to determine the resulting response characteristics and fuel consumption of the control motors.

GENERAL EQUATIONS OF MOTION

The general equations describing the motion of a satellite vehicle about its center of mass and including the effect of the gravitational gradient have been derived in Ref. 1 as follows

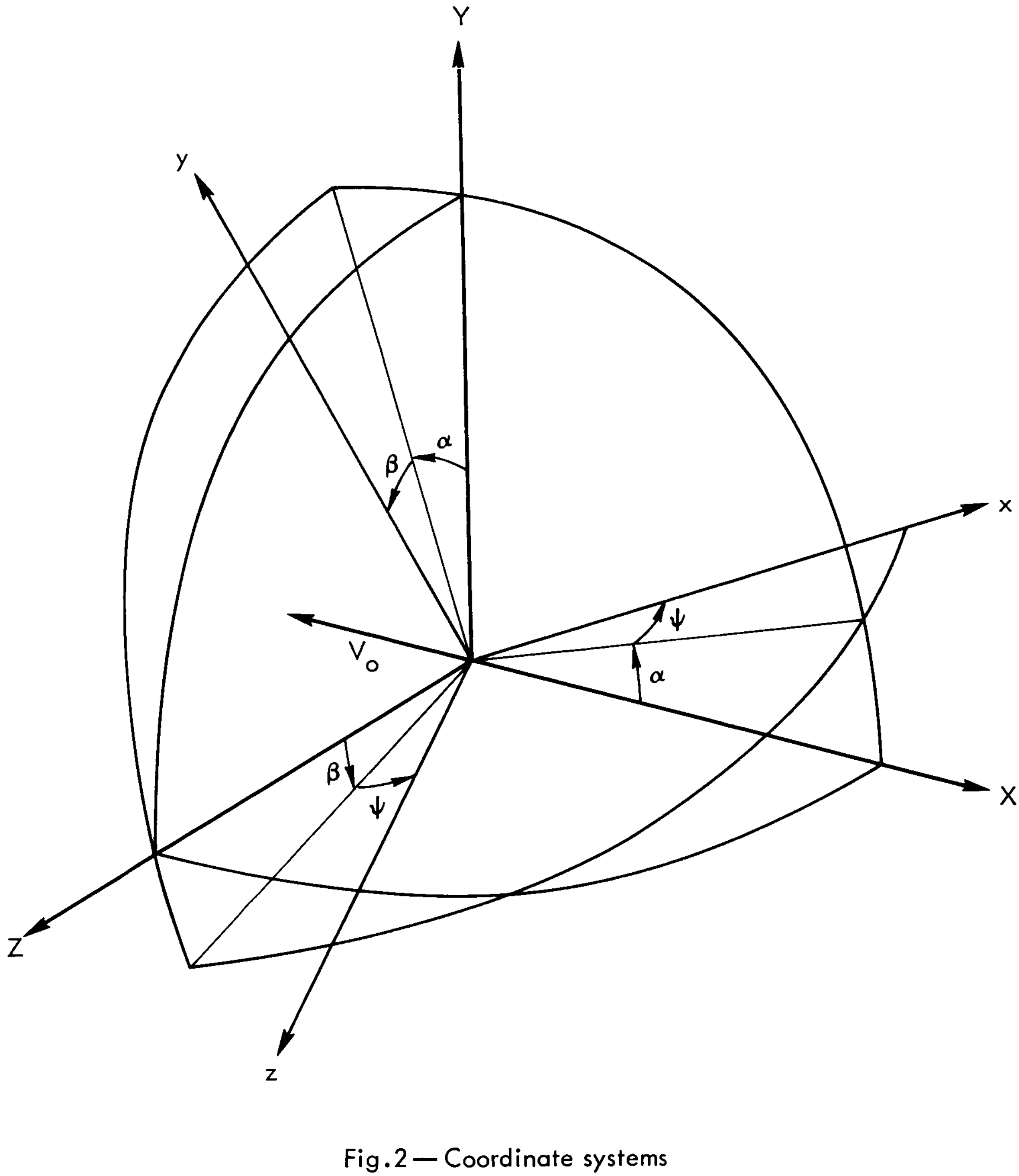

Fig.2 — Coordinate systems

$$\frac{d\omega_x}{dt} + \left(\frac{I_z - I_y}{I_x}\right)\omega_y\omega_z = \frac{3g_oR_o^2}{r^3}\left(\frac{I_z - I_y}{I_x}\right)b_yc_y + \frac{M_{cx}}{I_x} \tag{1}$$

$$\frac{d\omega_y}{dt} + \left(\frac{I_x - I_z}{I_y}\right)\omega_z\omega_x = \frac{3g_oR_o^2}{r^3}\left(\frac{I_x - I_z}{I_y}\right)a_yc_y + \frac{M_{cy}}{I_y} \tag{2}$$

$$\frac{d\omega_z}{dt} + \left(\frac{I_y - I_x}{I_z}\right)\omega_x\omega_y = \frac{3g_oR_o^2}{r^3}\left(\frac{I_y - I_x}{I_z}\right)b_ya_y + \frac{M_{cz}}{I_z} \tag{3}$$

where ω_x, ω_y, ω_z are the components of the vehicle angular velocity and are given by

$$\omega_x = -(\dot{\alpha} + \dot{\theta})\cos\beta\sin\psi + \dot{\beta}\cos\psi \tag{4}$$

$$\omega_y = (\dot{\alpha} + \dot{\theta})\sin\beta + \dot{\psi} \tag{5}$$

$$\omega_z = (\dot{\alpha} + \dot{\theta})\cos\beta\cos\psi + \dot{\beta}\sin\psi \tag{6}$$

and a_y, b_y, c_y are the direction cosines of the Y axis in the x, y, z system. These are given by

$$a_y = \sin\alpha\cos\psi + \cos\alpha\sin\beta\sin\psi \tag{7}$$

$$b_y = \cos\alpha\cos\beta \tag{8}$$

$$c_y = \sin\alpha\sin\psi - \cos\alpha\sin\beta\cos\psi \tag{9}$$

The quantities I_x, I_y and I_z represent the moments of inertia of the vehicle in roll, yaw and pitch respectively; while M_{cx}, M_{cy} and M_{cz} are the corresponding control moments. In addition, R_o is the radius of the earth, g_o is the gravitational acceleration at the earth's surface and r is the radial distance from the earth's center to the vehicle.

LINEARIZED EQUATIONS OF MOTION

In view of the fact that at synchronous altitude the earth subtends an angle of 17.4°, the required values of α, β and ψ for terrestrial targets will be small. As a result, the above equations can be simplified as follows.

For a circular orbit the orbital angular rate is constant and given by

$$\dot{\theta}^2 = \omega_o^2 = \frac{g_o R_o^2}{r^3} \tag{10}$$

Also, for small values of the angles α, β and ψ, the angular velocity components ω_x, ω_y and ω_z become

$$\omega_x = -\omega_o\psi + \dot{\beta} \tag{11}$$

$$\omega_y = \omega_o\beta + \dot{\psi} \tag{12}$$

$$\omega_z = \omega_o + \dot{\alpha} \tag{13}$$

while the direction cosines a_y, b_y and c_y are given by

$$a_y = \alpha \tag{14}$$

$$b_y = 1 \tag{15}$$

$$c_y = -\beta \tag{16}$$

Substitution of Eqs. (10-16) in Eqs. (1-3) gives

$$\ddot{\beta} + 4\omega_o^2\left(\frac{I_z - I_y}{I_x}\right)\beta + \left(\frac{I_z - I_y - I_x}{I_x}\right)\omega_o\dot{\psi} = \frac{M_{cx}}{I_x} \tag{17}$$

$$\ddot{\psi} + \omega_o^2\left(\frac{I_z - I_x}{I_y}\right)\psi - \left(\frac{I_z - I_y - I_x}{I_y}\right)\omega_o\dot{\beta} = \frac{M_{cy}}{I_y} \tag{18}$$

$$\ddot{\alpha} + 3\omega_o^2 \left(\frac{I_x - I_y}{I_z}\right)\alpha = \frac{M_{cz}}{I_z} \tag{19}$$

Since the configuration is planar and circular, the following relations exist between the moments of inertia

$$I_x = I_z = \frac{1}{2} I_y \tag{20}$$

This further simplifies Eqs. (17-19) to the form

$$\ddot{\beta} - 4\omega_o^2\beta - 2\omega_o\dot{\psi} = \frac{M_{cx}}{I_x} \tag{21}$$

$$\ddot{\psi} + \omega_o\dot{\beta} = \frac{M_{cy}}{I_y} \tag{22}$$

$$\ddot{\alpha} - 3\omega_o^2\alpha = \frac{M_{cz}}{I_z} \tag{23}$$

It is seen that basically Eqs. (21)-(23) represent the behavior of a moment of inertia under the influence of a control moment M_c. However, in the case of both pitch and roll there is a small destabilizing term. Also, in the case of roll and yaw the motions are coupled through the orbital angular rate ω_o. Since the coupling is weak, it is possible to specify the pitch, roll and yaw control moments as follows

$$M_{cx} = - k_{1x}(\beta - \beta_i) - k_{2x}\dot{\beta} \tag{24}$$

$$M_{cy} = - k_{1y}\psi - k_{2y}\dot{\psi} \tag{25}$$

$$M_{cz} = - k_{1z}(\alpha - \alpha_i) - k_{2z}\dot{\alpha} \tag{26}$$

where α_i and β_i are the input values of the pitch and roll angles.

The corresponding value of ψ_i is zero since there is no reason to drive the system in yaw.

Substitution of Eqs. (24-26) into Eqs. (21-23) gives

$$\ddot{\beta} + 2\omega_n\dot{\beta} + \omega_n^2\beta - 2\omega_o\dot{\psi} = (\omega_n^2 + 4\omega_o^2)\ \beta_i \tag{27}$$

$$\ddot{\psi} + 2\omega_n\dot{\psi} + \omega_n^2\psi + \omega_o\dot{\beta} = 0 \tag{28}$$

$$\ddot{\alpha} + 2\omega_n\dot{\alpha} + \omega_n^2\alpha = (\omega_n^2 + 3\omega_o^2)\ \alpha_i \tag{29}$$

where

$$\omega_n^2 = \frac{k_{1x}}{I_x} - 4\omega_o^2 = \frac{k_{1y}}{I_y} = \frac{k_{1z}}{I_z} - 3\omega_o^2 \tag{30}$$

$$2\omega_n = \frac{k_{2x}}{I_x} = \frac{k_{2y}}{I_y} = \frac{k_{2z}}{I_z} \tag{31}$$

This particular choice of control constants results in a critically damped response in pitch, roll and yaw.

It is seen that Eq. (29) describing the pitch motion is completely uncoupled whereas Eqs. (27) and (28) describing the roll and yaw motions are coupled. The solutions of these equations are discussed in the next two sections.

PITCH RESPONSE

In this section the solution for the pitch response to a step change in α_i and also a sinusoidal variation of α_i are determined as well as the propulsion requirements for these responses.

Step Input

If the desired pitch angle α_i is given by the relation

$$\alpha_i = 0 \text{ for } t < 0$$
$$= \alpha_o \text{ for } t > 0 \tag{32}$$

then the solution of Eq. (29) is of the form

$$\alpha = \frac{\omega_n^2 + 3\omega_o^2}{\omega_n^2} \alpha_o \left[1 - e^{-\omega_n t} (1 + \omega_n t)\right] \tag{33}$$

which is the characteristic critically damped response to a step input.

If a system time constant T_o is defined as

$$T_o = \frac{2\pi}{\omega_n} \tag{34}$$

then by means of Eq. (33) it can be shown that T_o is the time it takes for α to reach 98.64 per cent of its final value in response to a step input. Figure 3 is a plot of α as a function of t/T_o.

If T_o is small compared to the orbital period (24 hours) then Eq. (33) becomes

$$\alpha = \alpha_o \left[1 - e^{-\omega_n t} (1 + \omega_n t)\right] \tag{35}$$

The required control moment can be determined by substituting Eq. (33) into Eq. (23) to give

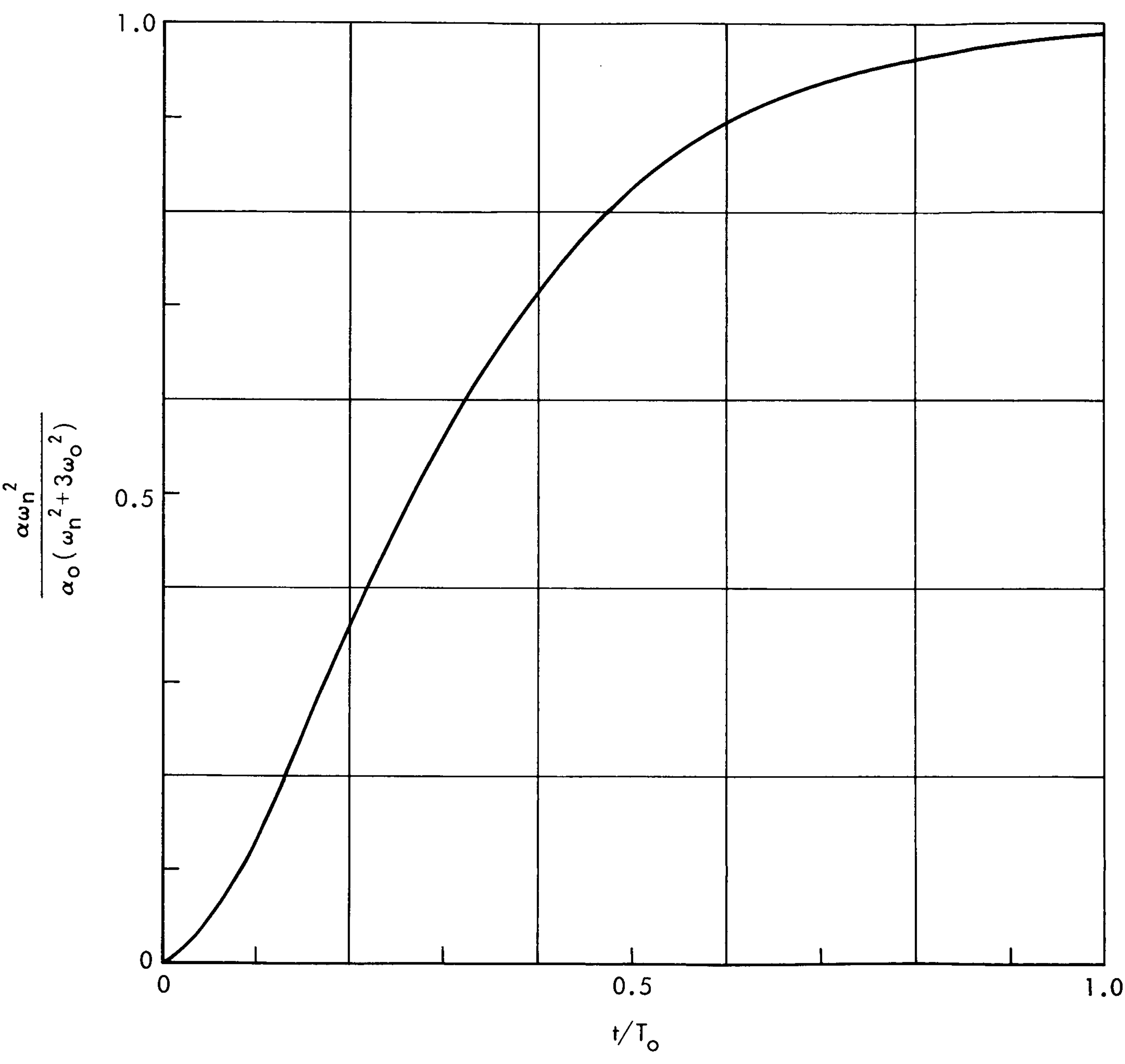

Fig.3— Step response in pitch attitude

$$\frac{M_{cz}}{I_z} = \frac{\omega_n^2 + 3\omega_o^2}{\omega_n^2} \alpha_o \left[- 3\omega_o^2 + e^{-\omega_n t} \left[(\omega_n^2 + 3\omega_o^2) - (\omega_n^2 - 3\omega_o^2)\ \omega_n t \right] \right] \tag{36}$$

and if T_o is small (i.e., $\omega_n \gg \omega_o$) then

$$\frac{M_{cz}}{I_z} = \alpha_o \left[- 3\omega_o^2 + \omega_n^2\ e^{-\omega_n t}\ (1 - \omega_n t) \right] \tag{37}$$

Equation (37) shows that the control moment is composed of a steady-state term and a transient term . These two parts are considered separately as follows.

<u>Transient Control Moment</u>. From Eq. (37) the transient control moment is given by

$$\left(\frac{M_{cz}}{I_z} \right)_{\text{Trans.}} = \alpha_o \omega_n^2 e^{-\omega_n t}\ (1 - \omega_n t) \tag{38}$$

If this moment is supplied by a rocket motor of thrust T, located on the periphery of the flat plate, then

$$M_{cz} = Tr_o \tag{39}$$

where r_o is the radius of the plate.

The thrust of the motor is given by

$$T = \frac{\alpha_o I_z\ \omega_n^2}{r_o}\ e^{-\omega_n t}\ (1 - \omega_n t) \tag{40}$$

The total impulse I is given by

$$I = \int_0^{\infty} |T| \, dt$$

$$= \frac{2\alpha_o I_z \omega_n}{e \, r_o} \tag{41}$$

If the flat plate has a uniform mass distribution, its moment of inertia in pitch is given by

$$I_z = \frac{1}{4} m_o r_o^2 \tag{42}$$

and the total impulse becomes

$$I = \frac{\pi \alpha_o m_o r_o}{e \, T_o} \tag{43}$$

If the fuel used in the rocket motor has a specific impulse of I_{sp} then the ratio of required fuel weight to total weight on orbit is given by

$$\left(\frac{m_F}{m_o}\right)_{Trans.} = \frac{\pi \alpha_o r_o}{e g_o T_o I_{sp}} \tag{44}$$

<u>Steady-State Control Moment</u>. The steady-state moment from Eq. (37) is

$$\left(\frac{M_{cz}}{I_z}\right)_{ss} = - 3\omega_o^2 \alpha_o \tag{45}$$

The required thrust is given by

$$T = - \frac{3 m_o r_o \omega_o^2 \alpha_o}{4} \tag{46}$$

and the total impulse is expressed as

$$I = \int_0^t |T|\, dt = \frac{3m_o r_o \omega_o^2 \alpha_o}{4} t \tag{47}$$

Finally, the mass of fuel required is given by

$$\left(\frac{m_F}{m_o}\right)_{ss} = \frac{3r_o \omega_o^2 \alpha_o}{4g_o I_{sp}} t \tag{48}$$

This represents a steady consumption of fuel needed to hold the vehicle at an angle α_o against the gravity gradient moment about the pitch axis.

Equation (48) is based on the assumption that α_o is a small angle. However, for deep space communication or radio astronomy applications, it may be necessary to consider large pitch angles. An examination of Eq. (3) shows that the control moment necessary to counteract the gravity gradient moment in pitch is

$$\frac{M_{cz}}{I_z} = - \frac{3g_o R_o^2}{r^3} \left(\frac{I_y - I_x}{I_z}\right) b_y a_y$$

$$= - 3\omega_o^2 \sin \alpha \cos \alpha \tag{49}$$

where both β and ψ are zero. The maximum value of M_{cz} occurs for α equal to 45^o in which case

$$\frac{M_{cz}}{I_z} = - \frac{3\omega_o^2}{2} \tag{50}$$

By the method used in obtaining Eq. (48), it can be shown that the steady-state fuel consumption is

$$\left(\frac{m_F}{m_o}\right)_{ss} = \frac{3r_o\,\omega_o^2\,t}{8g_o I_{sp}} \tag{51}$$

Maximum Acceleration. An examination of Eq. (38) shows that the maximum control moment occurs at time zero and is given by

$$\left(\frac{M_{cz}}{I_z}\right)_{max} = \alpha_o\omega_n^2 = \frac{d^2\alpha}{dt^2} \tag{52}$$

Thus, the maximum linear acceleration of the point of thrust application is expressed in g's as

$$n = \frac{r_o\alpha_o\,\omega_n^2}{g_o}$$

$$= \frac{4\pi^2 r_o\,\alpha_o}{g_o T_o^2} \tag{53}$$

Sinusoidal Input

For certain applications, it may be desirable to cause the plate to undergo a sinusoidal oscillation in pitch. In this case

$$\alpha_i = \alpha_o \sin \omega t \tag{54}$$

and the resulting steady-state solution of Eq. (29) is

$$\alpha = \frac{\alpha_o(\omega_n^2 + 3\omega_o^2)}{\omega_n^2 + \omega^2} \sin(\omega t - \varphi) \tag{55}$$

where

$$\tan \varphi = \frac{2\omega_n\omega}{\omega_n^2 - \omega^2} \tag{56}$$

and ω is the driving frequency.

Substitution of Eq. (55) into Eq. (23) gives the control moment required as

$$\frac{M_{cz}}{I_z} = - \frac{\alpha_o(\omega_n^2 + 3\omega_o^2)\ (\omega^2 + 3\omega_o^2)}{\omega_n^2 + \omega^2} \sin\ (\omega t - \varphi) \tag{57}$$

If ω_n is large compared to both the orbital angular rate ω_o and the driving frequency ω, then

$$\frac{M_{cz}}{I_z} = - \alpha_o(\omega^2 + 3\omega_o^2)\ \sin\ (\omega t - \varphi) \tag{58}$$

As before, the required rocket thrust is determined as

$$T = - \frac{\alpha_o\ m_o r_o}{4} (\omega^2 + 3\omega_o^2)\ \sin\ (\omega t - \varphi) \tag{59}$$

The total impulse per cycle is obtained from the expression

$$I = \int_{\frac{\varphi}{\omega}}^{\frac{2\pi + \varphi}{\omega}} |T|\ dt$$

$$= - 2 \int_{\frac{\varphi}{\omega}}^{\frac{\pi + \varphi}{\omega}} T\ dt$$

$$= \frac{\alpha_o\ m_o r_o(\omega^2 + 3\omega_o^2)}{\omega} \tag{60}$$

from which the required fuel consumption per cycle is determined as

$$\frac{m_F}{m_o} = \frac{\alpha_o r_o (\omega^2 + 3\omega_o^2)}{g_o I_{sp} \omega} \text{ per cycle} \tag{61}$$

YAW-ROLL RESPONSE

As has already been indicated, Eqs. (21) and (22), which determine the roll and yaw behavior of the vehicle, are coupled. Thus, although roll and pitch determine the orientation of the normal of the plate, it is not possible to control the roll attitude without inducing motion in yaw. If this yawing motion is not controlled and reduced to zero, the identification of the appropriate pitch and roll control motors becomes difficult. Thus, in determining the fuel requirement for roll control, it is necessary to include the fuel used in damping out the induced yaw motion.

Step Input

If the desired roll angle β_i is given by the relation

$$\begin{aligned} \beta_i &= 0 \qquad t < 0 \\ &= \beta_o \qquad t > 0 \end{aligned}$$

then the simultaneous solution of Eqs. (21) and (22) gives

$$\beta = \frac{\omega_n^2 + 4\omega_o^2}{\omega_n^2} \beta_o \left[1 - e^{-\omega_n t} (1 + \omega_n t) \right] \tag{62}$$

and

$$\psi = - \frac{(\omega_n^2 + 4\omega_o^2)}{6} \beta_o \omega_o t^3 \, e^{-\omega_n t} \tag{63}$$

where the effect of cross coupling on the characteristic frequencies is neglected.

Substitution of these solutions into Eqs. (24) and (25) gives the following control moment expressions

$$\frac{M_{cx}}{I_x} = \beta_o \left[- 4\omega_o^2 + \omega_n^2 e^{-\omega_n t} (1 - \omega_n t) \right] \tag{64}$$

$$\frac{M_{cy}}{I_y} = - \frac{\beta_o \omega_o \omega_n^3 t^2 (6 - \omega_n t) e^{-\omega_n t}}{6} \tag{65}$$

It is seen that the roll control moment has both a steady-state and a transient component while the yaw control moment has only a transient term. As in the pitch response, these will be treated separately.

<u>Transient Control Moment</u>. As before, the fuel requirement for the transient terms of Eqs. (64) and (65) can be expressed as follows

$$\left(\frac{m_F}{m_o} \right)_{Trans.} = \frac{\beta_o r_o}{g_o I_{sp}} \left[\frac{\pi}{eT_o} + \frac{\omega_o}{2} \right] \tag{66}$$

where the first term arises from the roll control moment while the last is from the induced yaw control and terms of the order of e^{-6} inside the brackets have been neglected.

<u>Steady-State Control Moment</u>. The steady-state control moment in the roll equation gives rise to a fuel requirement expressed as

$$\left(\frac{m_F}{m_o} \right)_{ss} = \frac{r_o \omega_o^2 \beta_o}{g_o I_{sp}} t \tag{67}$$

While Eq. (67) applies for small values of β_o, the corresponding relation for large angles is obtained in a manner similar to that used in the pitch case. The roll control moment is given by

$$\frac{M_{cx}}{I_x} = -4\omega_o^2 \sin\beta \cos\beta \tag{68}$$

where both α and ψ are zero. As before, the maximum moment occurs for β equal to 45^o in which case the steady-state fuel requirement is

$$\left(\frac{m_F}{m_o}\right)_{ss} = \frac{r_o \omega_o^2 t}{2 g_o I_{sp}} \tag{69}$$

Sinusoidal Input

As in the case of the pitch motion, it may be desirable to drive the system sinusoidally in roll. In this case

$$\beta_i = \beta_o \sin \omega t \tag{70}$$

and the resulting steady-state solution of Eqs. (27) and (28) has the form

$$\beta = \frac{\omega_n^2 + 4\omega_o^2}{\omega_n^2 + \omega^2} \beta_o \sin(\omega t - \varphi) \tag{71}$$

and

$$\psi = \frac{(\omega_n^2 + 4\omega_o^2)\ \beta_o \omega_o \omega}{(\omega_n^2 + \omega^2)^2} \cos(\omega t - \delta) \tag{72}$$

where

$$\tan \varphi = \frac{2\omega\omega_n}{\omega_n^2 - \omega^2} \tag{73}$$

$$\tan \delta = \frac{4\omega\omega_n(\omega_n^2 - \omega^2)}{\omega^4 - 6\omega_n^2\omega^2 + \omega_n^4} \tag{74}$$

The fuel requirement associated with the roll control system can be written down by analogy with Eq. (61) as follows

$$\frac{m_F}{m_o} = \frac{\beta_o r_o(\omega^2 + 4\omega_o^2)}{g_o I_{sp}\omega} \text{ per cycle} \tag{75}$$

while the fuel requirement for the associated yaw control response can be determined as

$$\frac{m_F}{m_o} = \frac{2\beta_o r_o \omega_o}{g_o I_{sp}} \text{ per cycle} \tag{76}$$

where ω_n^2 is large compared to either ω^2 or ω_o^2. Thus, the total fuel requirement for a sinusoidal motion in roll is given by the sum of Eqs. (75) and (76) as

$$\frac{m_F}{m_o} = \frac{\beta_o r_o}{g_o I_{sp}} \left[\frac{\omega^2 + 2\omega\omega_o + 4\omega_o^2}{\omega} \right] \text{ per cycle} \tag{77}$$

BANG-BANG CONTROL

In the analysis presented thus far, it has been assumed that the control motors have a continuously variable thrust determined by the control laws specified in Eqs. (24-26). However, it is entirely possible to mechanize a control system in which the control motors of Fig. 1 have a fixed value of thrust and the magnitude of the applied impulse is determined by the duration. As an example, consider the motion of the vehicle in pitch neglecting the gravity gradient term. Equation (23) becomes

$$\frac{d^2\alpha}{dt^2} = \frac{M_{cz}}{I_z} \tag{78}$$

If the control moment is given by

$$\begin{aligned} M_{cz} &= + M_o \quad (0 < t < \tfrac{T_o}{2}) \\ M_{cz} &= - M_o \quad (\tfrac{T_o}{2} < t < T_o) \\ M_{cz} &= \quad 0 \quad\quad (t > T_o) \end{aligned} \tag{79}$$

it can be shown that the solution of Eq. (78) is

$$I_z\alpha = \frac{M_o t^2}{2} \qquad\qquad (0 < t < \frac{T_o}{2}) \tag{80}$$

$$\begin{aligned} I_z\alpha = \frac{M_o T_o}{2}(t - \frac{T_o}{2}) - \frac{M_o}{2}(t - \frac{T_o}{2})^2 \\ + \frac{M_o T_o^2}{8} \qquad\qquad (\frac{T_o}{2} < t < T_o) \end{aligned} \tag{81}$$

Thus the resulting pitch deflection after the application of the double pulse is given by

$$I_z \alpha_o = \frac{M_o T_o^2}{4} \tag{82}$$

The steady thrust required to produce a deflection α_o in time T_o is expressed as

$$T = \frac{4 I_z \alpha_o}{r_o T_o^2}$$

$$= \frac{m_o r_o \alpha_o}{T_o^2} \tag{83}$$

while the total impulse is

$$I = T T_o = \frac{m_o r_o \alpha_o}{T_o} \tag{84}$$

and the required fuel is determined as

$$\frac{m_F}{m_o} = \frac{r_o \alpha_o}{g_o I_{sp} T_o} \tag{85}$$

A comparison of Eq. (85) with Eq. (44) shows that the variable thrust system requires slightly more fuel by a factor of π/e or 1.155.

However, it can be shown that the maximum peripheral acceleration in this case is given by

$$n = \frac{4 \alpha_o r_o}{g_o T_o^2} \quad (g's) \tag{86}$$

A comparison of Eq. (86) with Eq. (53) shows that the variable thrust system applies higher peak accelerations by a factor of π^2 or about 10.

III. RESULTS

This section presents the numerical results based on the analysis presented in the previous section. These results are in the form of the fuel requirements for various modes of attitude control.

STEP CHANGE IN ATTITUDE

The fuel requirement for step changes in pitch and roll are given by Eqs. (44) and (66) respectively. An examination of these two relations shows that if the system time constant T_o is small compared to the 24-hour orbital period, the two expressions are identical and can be expressed in the form

$$\frac{m_F}{m_o} = 6.264 \times 10^{-4} \, \frac{A_o(\text{deg}) \; r_o(\text{ft})}{T_o(\text{sec}) \; I_{sp}(\text{sec})} \tag{87}$$

where A_o is the step amplitude α_o or β_o. Figure 4 presents a plot of Eq. (87) as a function of the response time T_o.

As an example, suppose that a flat plate of radius r_o equal to 500 ft is required to make an attitude change of A_o equal to 17°, in a time T_o equal to 300 sec using a control rocket fuel with a specific impulse I_{sp} equal to 70 sec. From Fig. 4 it is seen that

$$\frac{m_F}{m_o}\left(\frac{I_{sp}}{A_o r_o}\right) = 2.088 \times 10^{-6} \tag{88}$$

or

$$\frac{m_F}{m_o} = 2.538 \times 10^{-4} \tag{89}$$

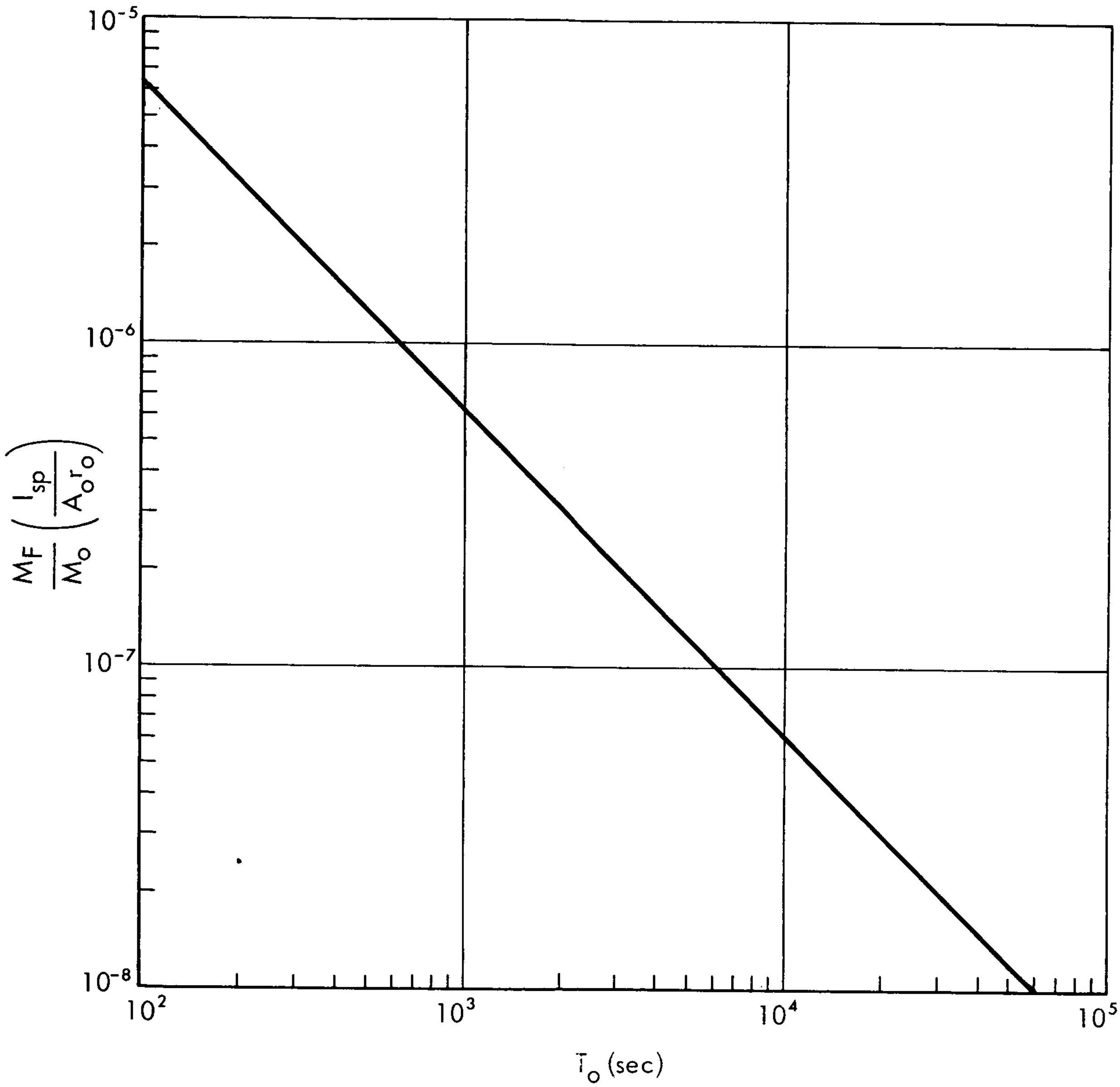

Fig. 4— Fuel requirement for step change in pitch or roll attitude

The above case represents a change in attitude equal to the total angle subtended by the earth at synchronous altitude utilizing a fuel comparable to either cold gas or a subliming solid.

While the mass fraction required for one such step change seems small, the total requirement would depend on the frequency at which such step changes might be required.

From Eq. (53) the maximum peripheral acceleration during this step response can be determined as 2.021×10^{-3} g's, while the corresponding maximum acceleration using the bang-bang control system is determined from Eq. (86) as 2.048×10^{-4} g's.

STEADY-STATE DEFLECTION

Equations (48) and (67) give the fuel requirements for maintaining steady deflections in pitch and roll respectively and can be expressed as

Pitch

$$\left(\frac{m_F}{m_o}\right)_{ss} = 1.858 \times 10^{-7} \; \frac{r_o(\text{ft}) \; \alpha_o(\text{deg}) \; t(\text{days})}{I_{sp}(\text{sec})} \tag{90}$$

Roll

$$\left(\frac{m_F}{m_o}\right)_{ss} = 2.477 \times 10^{-7} \; \frac{r_o(\text{ft}) \; \beta_o(\text{deg}) \; t(\text{days})}{I_{sp}(\text{sec})} \tag{91}$$

Figure 5 is a plot of Eqs. (90) and (91) where A_o represents either α_o or β_o.

As an example, suppose a plate of radius r_o equal to 500 ft is required to hold a pitch attitude of 8^o for one year using a control rocket fuel with a specific impulse of 70 sec. From Fig. 5 it is seen that

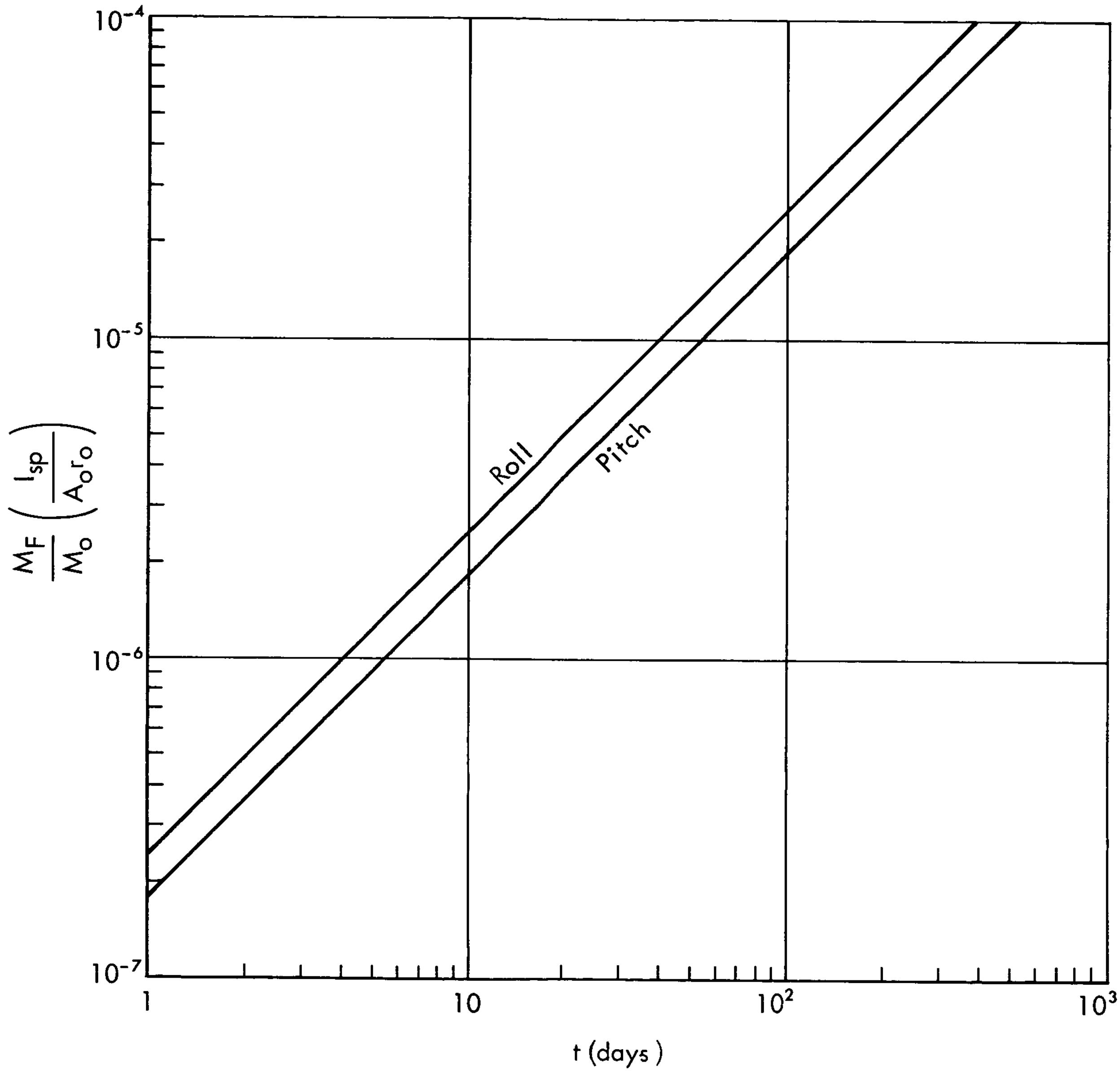

Fig. 5—Fuel requirement for steady-state deflection in pitch or roll attitude

$$\left(\frac{m_F}{m_o}\right)\left(\frac{I_{sp}}{r_o A_o}\right) = 6.782 \times 10^{-5} \tag{92}$$

or

$$\frac{m_F}{m_o} = 3.876 \times 10^{-3} \tag{93}$$

The 8^o deflection in pitch is approximately the maximum deflection from the vertical required for any terrestrial target from synchronous altitude, and such a deflection requires less than .5 per cent of total vehicle weight for the fuel to maintain this attitude for a year.

As pointed out in the previous section, the maximum steady-state control moment required to counteract the gravity gradient moment occurs for a deflection angle of 45^o in either pitch or roll. The fuel requirement is given by Eqs. (51) and (69) which can be rewritten as

Pitch

$$\left(\frac{m_F}{m_o}\right)_{45^o} = 5.321 \times 10^{-6} \; \frac{r_o(\text{ft}) \; t(\text{days})}{I_{sp}(\text{sec})} \tag{94}$$

Roll

$$\left(\frac{m_F}{m_o}\right)_{45^o} = 7.093 \times 10^{-6} \; \frac{r_o(\text{ft}) \; t(\text{days})}{I_{sp}(\text{sec})} \tag{95}$$

Figure 6 is a plot of Eqs. (94) and (95).

As an example, the yearly amount of fuel required to maintain a 500-ft radius plate at a 45^o pitch attitude if I_{sp} is 70 sec is obtained from Fig. 6 as follows

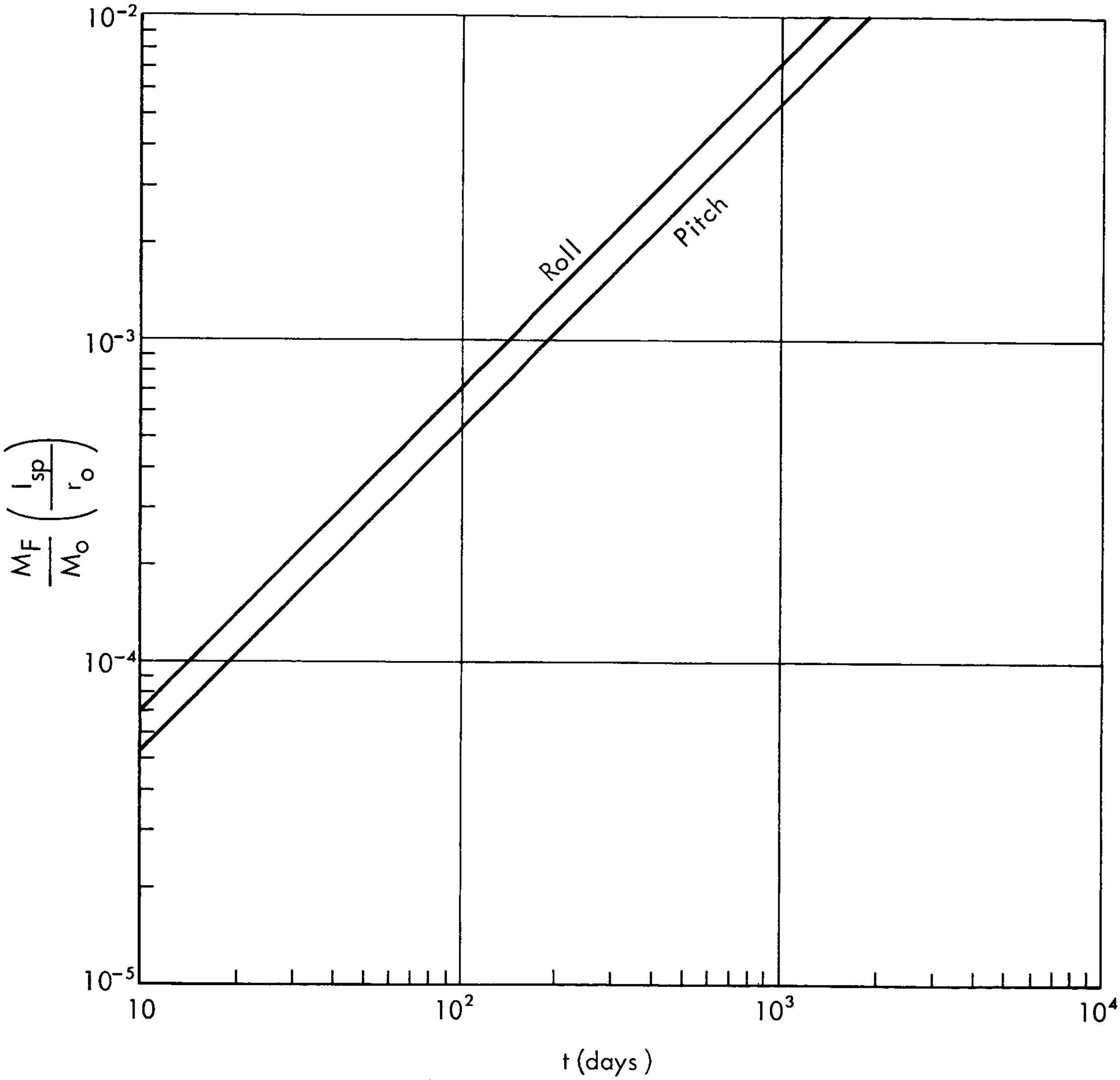

Fig.6—Fuel requirement to maintain a 45° pitch or roll attitude

$$\left(\frac{m_F}{m_o}\right)\left(\frac{I_{sp}}{r_o}\right) = 1.942 \times 10^{-3} \tag{96}$$

or

$$\frac{m_F}{m_o} = 1.387 \times 10^{-2} \tag{97}$$

Thus, about 1.4 per cent of the total vehicle weight would be needed per year to maintain its attitude against the maximum gravity gradient moment in pitch.

SINUSOIDAL ATTITUDE VARIATION

The fuel requirements to maintain sinusoidal oscillations of the plate in pitch and in yaw are specified in Eqs. (61) and (77) respectively. If the driving frequency ω is large compared to the orbital angular rate ω_o, these two equations reduce to the form

$$\frac{m_F}{m_o} = \frac{A_o r_o \omega}{g_o I_{sp}} \text{ per cycle} \tag{98}$$

where A_o represents either α_o or β_o. Equation (98) can be modified to give the fuel requirement for a time t as follows

$$\frac{m_F}{m_o} = \frac{2\pi A_o r_o f^2}{g_o I_{sp}} t \tag{99}$$

or

$$\frac{m_F}{m_o} = 3.942 \times 10^{-8} \frac{A_o(\text{deg})\ r_o(\text{ft}) \left[f(\text{cycles/day})\right]^2 \ t(\text{days})}{I_{sp}(\text{sec})} \tag{100}$$

Figure 7 is a plot of Eq. (100) using f, the driving frequency, in cycles/day as the independent variable.

As an example, determine the daily fuel requirement for an 8^o pitch oscillation at 100 cycles per day for a plate of radius equal to 500 ft and a specific impulse of 70 sec. From Fig. 7 it is seen that

$$\frac{m_F}{m_o}\left(\frac{I_{sp}}{A_o r_o t}\right) = 3.942 \times 10^{-4} \tag{101}$$

or

$$\frac{m_F}{m_o} = 2.253 \times 10^{-2} \tag{102}$$

At this rate of fuel expenditure the vehicle would be reduced to half of its initial weight in about 30 days. Thus, the use of this type of system for mechanical scanning at this rate does not appear to be feasible.

ORBITAL INCLINATION COMPENSATION

If the reflecting plate is on a synchronous orbit which is inclined at an angle i_o to the equatorial plane, its ground trace describes the familiar figure eight pattern on the ground. It is of interest to determine the amount of fuel required to keep the plate pointing at a particular point on the earth. For example, suppose the equatorial crossing point is selected as the target to be tracked. Under these conditions the required pitch and roll angles are given by the relations

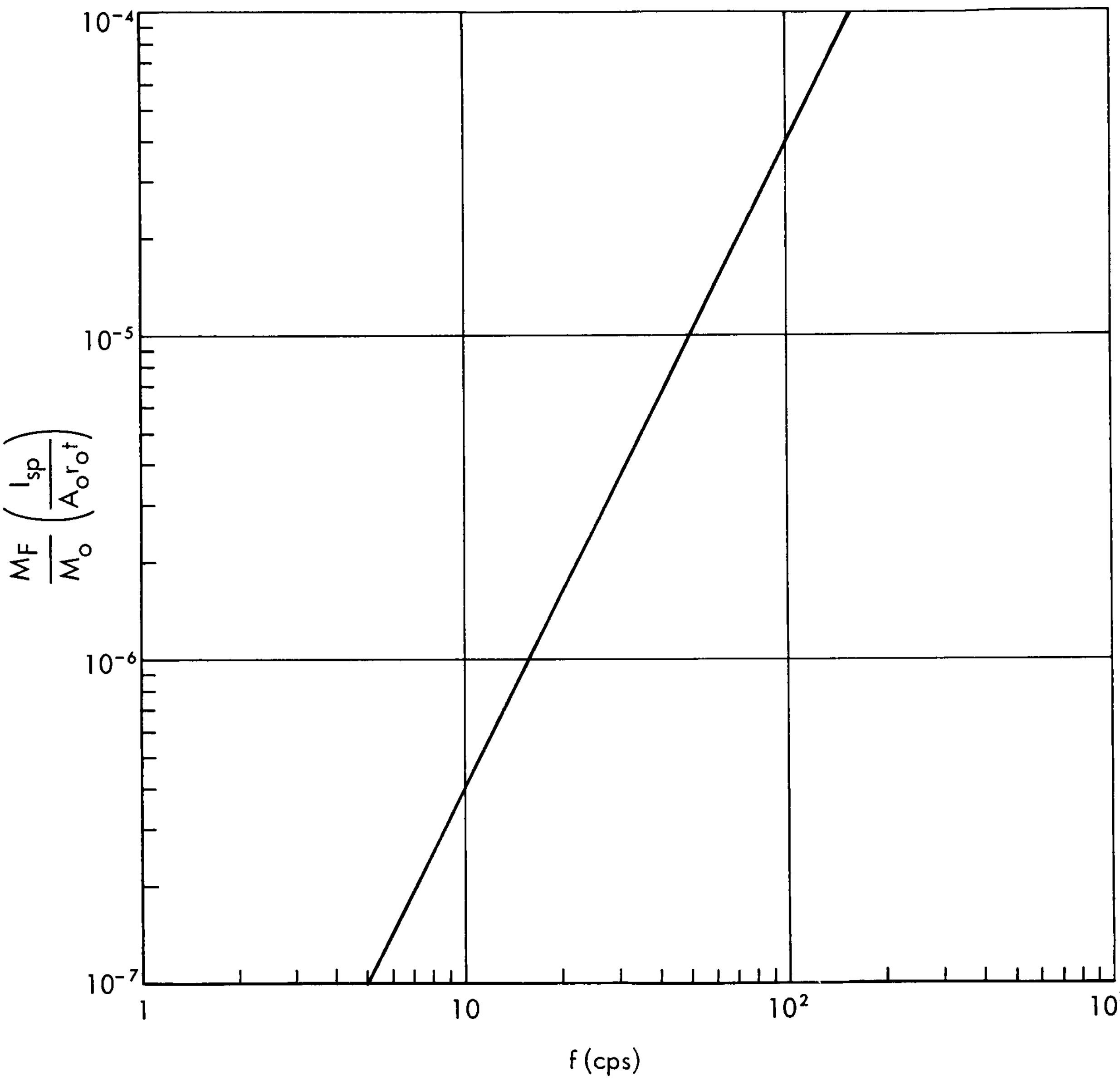

Fig. 7— Fuel requirement for driven oscillations in pitch or roll attitude

$$\alpha_i = - \alpha_o \sin 2\omega_o t \tag{103}$$

$$\beta_i = \beta_o \sin \omega_o t \tag{104}$$

where the pitch and roll amplitudes α_o and β_o are given to a good approximation for values of i_o less than about 45° by

$$\alpha_o = \frac{R_o(1 - \cos i_o)}{2(r_c - R_o)} \tag{105}$$

$$\beta_o = \frac{R_o \sin i_o}{r_c - R_o} \tag{106}$$

Substitution of these conditions into Eqs. (61) and (77) gives the total fuel requirement per day as

$$\frac{m_F}{m_o} = \frac{7 r_o \omega_o R_o}{2(r_c - R_o)\, g_o I_{sp}} (1 - \cos i_o + 2 \sin i_o) \tag{107}$$

As an example, if a 500-ft radius plate is on an orbit with a 30° inclination and with an I_{sp} of 70 sec, the yearly fuel requirement is given by

$$\frac{m_F}{m_o} = .452 \times 10^{-2} \tag{108}$$

or a little less than half of one percent of the vehicle weight.

ORBITAL ECCENTRICITY COMPENSATION

If a vehicle is on an equatorial orbit with a synchronous period but with a small eccentricity ϵ, then the subsatellite point on the equator has a central angle given by

$$\Delta\theta = 2\epsilon \sin \omega_o t \tag{109}$$

If the flat plate on orbit is required to track the mean subsatellite point on the earth, the required pitch angle variation is given by

$$\alpha_i = \frac{R_o}{r_c - R_o} \Delta\theta$$

$$= \frac{2\epsilon R_o}{r_c - R_o} \sin \omega_o t \tag{110}$$

The fuel requirement per day for the motion specified by Eq. (110) can be determined from Eq. (61) as

$$\frac{m_F}{m_o} = \frac{2r_o \omega_o R_o \epsilon}{g_o I_{sp}(r_c - R_o)} \tag{111}$$

For a 500-ft radius plate and a specific impulse of 70 sec, the yearly fuel requirement is found to be

$$\frac{m_F}{m_o} = .0023\epsilon \text{ per year} \tag{112}$$

which is neglibible for any reasonable eccentricity.

STATION-KEEPING PROPULSION

In addition to the various attitude control propulsion requirements considered above, it is also necessary to provide station-keeping propulsion to correct for the effects of the earth's equatorial ellipticity. In Ref. 2, the maximum propulsion requirement was indicated as 17 ft/sec per year of operation. However, since the publication of Ref. 2, analysis of the tracking data on Syncom 2 (Ref. 3) indicates that the J_{22} term in the earth's potential is less than had originally been assumed. A more realistic value of the maximum propulsion requirement if 6 ft/sec per year. The mass of fuel required for this purpose, assuming again a specific impulse of 70 sec, is found to be

$$\frac{m_F}{m_o} = \frac{\Delta V}{g_o I_{sp}}$$

$$= .00266 \text{ per year} \tag{113}$$

EFFECT OF RADIATION PRESSURE

It is obvious that for the idealized flat plate considered in this Memorandum there would be no moment due to solar radiation pressure since, by symmetry, the center of pressure and center of mass would be coincident. However, in an actual design, nonuniformities in either mass distribution or reflectivity of the surface can result in an offset between the center of pressure and the center of mass with a resulting moment applied to the vehicle.

It is of interest to determine the magnitude of this moment and the attitude control fuel requirement in order to counter this effect. The magnitude of the required control moment can be expressed as

$$M_c = \pi r_o^2 \, p_o \, \Delta r \tag{114}$$

where p_o is the magnitude of the radiation pressure and Δr is the offset between center of pressure and center of mass.

The thrust required to produce this moment is given by

$$T = \pi r_o \; p_o \; \Delta r \tag{115}$$

and in the same manner as in the body of the Memorandum the impulse required is determined as

$$I = \int_o^t Tdt = \pi r_o \; p_o \; \Delta rt \tag{116}$$

Finally, the mass of fuel required is expressed as

$$m_F = \frac{\pi r_o \; p_o \; \Delta rt}{I_{sp}} \tag{117}$$

In this case the fuel requirement is independent of the total mass on orbit and depends only on the dimension of the vehicle and its reflecting properites.

As a worst case, it is assumed that the solar radiation is normally incident and that the surface is a perfect reflector. Under these conditions

$$p_o = 1.865 \times 10^{-7} \text{ lbs/ft}^2$$

and Eq. (117) can be expressed as

$$m_F(\text{lbs}) = 5.062 \times 10^{-2} \; \frac{r_o(\text{ft}) \; \Delta r(\text{ft}) \; t(\text{days})}{I_{sp}(\text{sec})} \tag{118}$$

which is plotted in Fig. 8.

As an example, suppose a 500-ft radius plate has an offset of 1 ft between the center of mass and the center of pressure and a specific impulse of 70 sec. Determine the yearly fuel requirement to counteract the resulting radiation pressure moment. From Fig. 8 it is seen that

$$m_F\left(\frac{I_{sp}}{r_o t}\right) = 5.062 \times 10^{-2} \tag{119}$$

and as a result

$$m_F = 132.0 \text{ lbs/yr} \tag{120}$$

This fuel expenditure does not seem excessive for a vehicle of the size assumed.

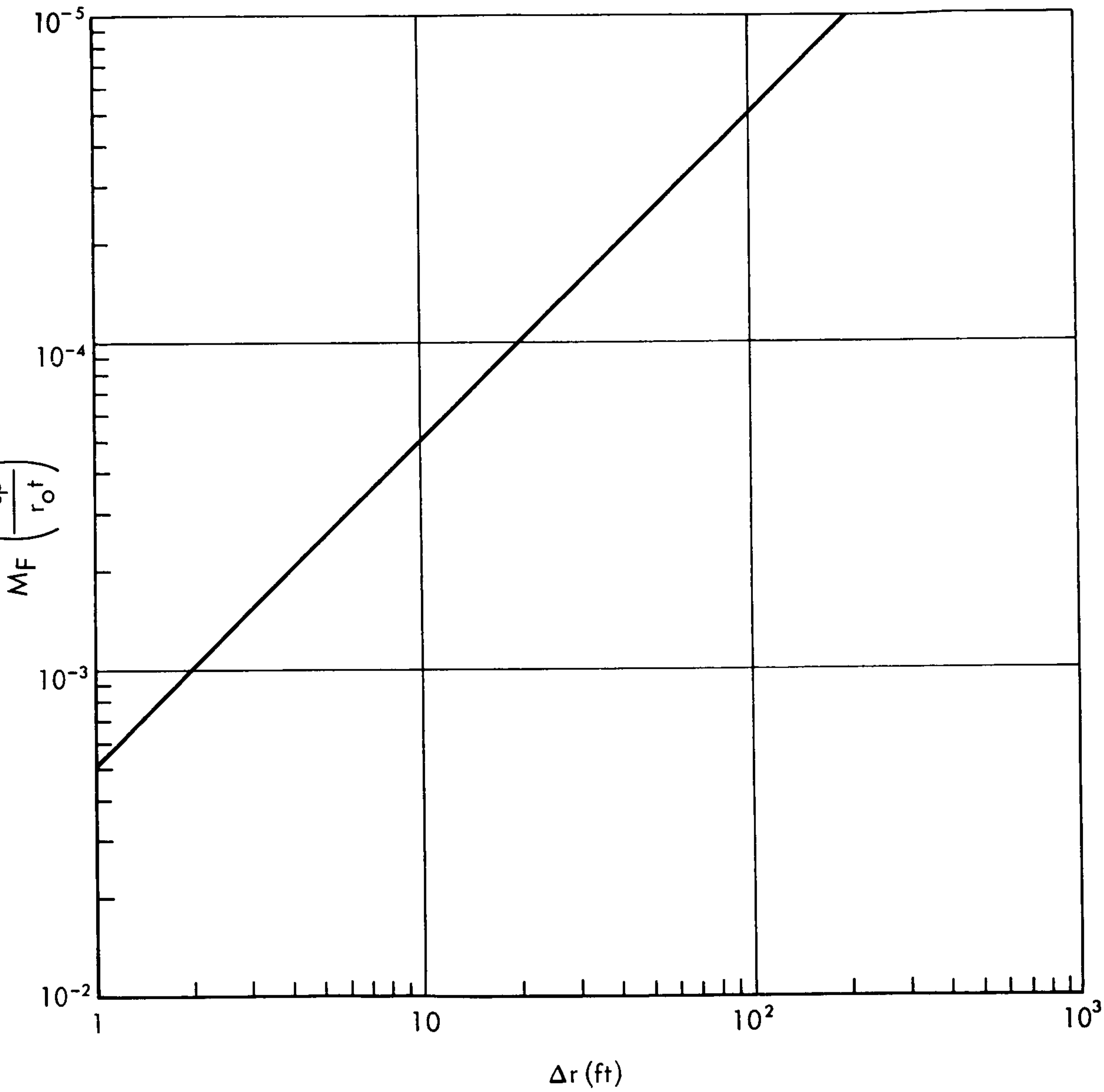

Fig. 8 — Fuel requirement for control of radiation pressure moment

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IV. DISCUSSION

In all of the expressions for fuel requirements determined in the previous section, it has been assumed that the plate has a uniform mass distribution. If, on the other hand, all of the mass is concentrated at the circumference of the plate, then all of the fuel requirements are doubled with the exception of that for station keeping.

An examination of the various controlled responses considered indicates that the resulting fuel requirements to maintain either a steady-state deflection or low-frequency (orbital period) oscillations are of the same order as the station-keeping fuel requirement. On the other hand, if the vehicle is required to undergo high-frequency (of the order of 100 cycles/day) oscillations in pitch and/or roll, the resulting fuel requirements can become excessive.

Obviously, the feasibility of controlling the attitude of such a vehicle is dependent on the size and mass distribution of the plate, the specific impulse of the fuel and the attitude variations required for the contemplated mission. Until these things are specified, the feasibility cannot be determined. However, the working formulas and graphs presented in Section III should be of assistance in arriving at a feasible design of such a control system.